## Problem description

We consider a production planning problem within an industrial symbiosis framework where two decentralized manufacturers have to plan their production over a planning horizon of T periods as illustrated in Figure 3. Manufacturer 1 (M1) operates a production unit that produces product 1 to meet a deterministic demand in each period t ∈ {1, 2,…,T} denoted dt1​. During production, a residue is generated in proportion to the quantity produced over time, Xt1​, with a proportionality factor r1. A binary variable Yt1 indicates whether production unit of M1 is running during period t. To meet environmental standards, residues must undergo mandatory pretreatment in specialized facilities, enabling their suitability for use as a raw material for Manufacturer 2 (M2). The quantity of by-products transferred from M1 to M2 in each period is denoted Bt ​and is traded at a price btr.

In each period, the second manufacturer M2 must select one of three operational modes: relying solely on virgin raw materials Vt sourced from and external supplier at price btv to produce a quantity Xt2 of its standard product, or produce Xth units of a hybrid product with a combination of α % by-products and (1-α)% virgin raw materials, or produce a pure eco-friendly product Xte using exclusively the by-product. M2 must satisfy a deterministic demand per period denoted dt2. We assume that the proportions of M2 customers, denoted by wh and we, demonstrate a strong sensitivity toward hybrid and pure eco-friendly products. The standard, hybrid and pure eco-friendly products of M2 are sold at a unit price of respectively at2, ath and ate.

In addition to supply costs, the conversion processes, including the production and preprocessing activities of M1 and M2, along with associated storage and landfilling operations, incur the following unit costs in each period t ∈ {1, 2, …, T}:

* Setup costs: these are associated with the production units of M1 and M2, as well as the treatment unit of M1, denoted by ft1, ft2 and ftr respectively.
* Production costs: costs incurred for producing product 1, products 2, and for preprocessing M1's residue are represented by pt1, pt2 and ptr respectively.
* Inventory holding costs: costs for storing product 1 (It1 quantity), products 2 (It2+ Ith +Ite quantity), and the preprocessed residue of M1 (Itr quantity) within limited-capacity inventories are denoted by ht1, ht2 and htr respectively.
* Landfilling costs: unit costs associated with disposing non-recyclable waste generated by the treatment unit of M1 (rrRt quantity) and the production unit of M2 (r2Xt2 + r2Xth +r2Xte quantity) are denoted by cL,t1 and cL,t2 respectively.

Furthermore, the proposed symbiotic supply chain accounts for carbon emissions generated per unit in each period from the following activities:

* Conversion activities: emissions resulting from production and residue preprocessing are denoted by emt1, emt2 and emtr.
* Storage: emissions associated with storing final products or residue are represented by emI,t1, emI,t2 and emI,tr.
* Landfilling: emissions from the disposal of non-recyclable waste generated by the treatment unit and the production unit of M2 are denoted by emL,t1 and emL,t2.

This problem statement showcases a practical example of a symbiotic supply chain in industries focused on sustainability and efficiency. For instance, steel slag is a solid by-product of the steel-making process, that must undergo essential pretreatment steps such as crushing, magnetic separation, and screening to comply with environmental standards and mitigate the leachability risks associated with heavy metals [25]. Once adequately treated, steel slag offers diverse applications, particularly in the construction sector. It can serve as a partial substitute for clay in brick production, promoting resource efficiency by reducing dependence on increasingly scarce virgin materials. Additionally, steel slag can be utilized as the sole raw material for manufacturing high-strength bricks, taking advantage of its intrinsic properties to produce durable and robust building materials. This adaptability not only supports sustainable practices by repurposing industrial waste but also drives the creation of innovative construction materials with enhanced mechanical performance.

A diagram of a company

Description automatically generated

Figure 3: Representation of the realistic by-products exchange flow within an industrial symbiosis framewor

## Mathematical modeling

Using notations given above, the profit functions and carbon emissions of each manufacturer can be modeled through the following straightforward formulation:

s.t.

Under carbon tax regulation, an additional cost paid in taxes for the carbon emissions released is subtracted from the manufacturers profit functions. Let be the monetary value charged as tax for every unit of carbon emissions released.